

Critical region for an Ising model coupled to causal dynamical triangulations

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Abstract

This paper extends results obtained by [15] for the annealed Ising model coupled to two-dimensional causal dynamical triangulations. We employ the Fortuin-Kasteleyn (FK) representation in order to determine a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve for the annealed model is possibly located. This is done by outlining a region where the model has a unique infinite-volume Gibbs measure, and a region where the finite-volume Gibbs measure does not have weak limit (in fact, does not exist if the volume is large enough). We also improve the region of subcritical behaviour of the model, and provide a better approximation of the free energy.

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1 Introduction. A review of related results

Models of planar random geometry appear in physics in the context of two-dimensional quantum gravity and provide an interplay between mathematical physics and probability theory.

Causal dynamical triangulation (CDT) and its predecessor dynamical triangulation (DT), introduced by Ambjørn and Loll (see [7]), constitute attempts to provide a meaning to the formal expressions appearing in the path integral quantization of gravity (see [5], [6] for an overview). The idea is to approximate the path integral by changing integration with respect to continuous random geometry by integration with respect to DTs or CDTs. As a result, the path integral with respect to continuous random geometries is replaced by a weighted sum over discrete geometries, where the weights are defined by the discrete form of the original action. In this paper we adopt the framework of CDTs coupled with an Ising spin system.

Putting a spin system on the collection of all causal triangulations is interpreted as a *coupling gravity with matter*. This type of coupling has been a subject of persistent interest in physics once the matrix model methods were successfully applied to the Ising model on random lattice with coordination number 4 (see [17] and [9] for details). In this article we discuss the Ising model coupled to causal triangulations in details, study the thermodynamic function known as free energy, and formally define the critical curve for the annealed model. This annealed model was first introduced by [3]. This model was also studied by [8], where the authors develop a systematic counting of embedded graphs and evaluate thermodynamic functions at high and low temperature expansion (see [4] and [15] for other progress made for the annealed model).

The main advantage of the CDTs over the DTs is that the ensemble of CDTs is more regular than the ensemble of the DTs, in the sense that does not permit spacial topology changes and leads to a fractal dimension $d_H = 2$ (see [7], [5], [10], [11]). On the other hand, the causality constraints still make it difficult to find an analytical solution of the Ising model coupled to CDTs, however some progress has been recently made about existence of Gibbs measures and phase transitions (see [15], [20] for details). Employing transfer matrix techniques and in particular the Krein-Rutman theory of positivity-preserving operators, [15] determines a region in the quadrant of parameters $\beta, \mu > 0$ where the finite-volume free energy has a thermodynamic limit. In this article the authors provide first approximations of the infinite-volume free energy, and conditions for existence of Gibbs measures. Furthermore, the authors prove that in the determined region the annealed

model presents a subcritical behaviour, i.e., there exists a unique Gibbs distribution for the model, and that with respect to this limit measure the average surface has a form of an infinite narrow tube, which is essentially one dimensional.

Computation of the critical curve (point) for models in statistical mechanics has always been a challenging problem, and since FK models were introduced by Fortuin and Kasteleyn in 1969 (see [12]), they have become an important tool in the study of phase transition for spin models. The goal of this article is to use the FK representation of the Ising model coupled to CDT in order to obtain lower and upper bounds for the critical curve. We employ the results of [15] to establish a region where the N-strip Gibbs measure has no weak limit as $N \rightarrow \infty$, where the boundary of this region provides a lower bound for the critical curve. Furthermore, we expand the region of subcritical behaviour of the model, and provide a better approximation to the free energy. The aforementioned FK representation utilizes a family of Poisson point processes and the Lie-Trotter product formula to interpret exponential sums of operators as random operator products. This representation was originally derived in [2] (see [1], [16] for an overview).

Results presented in this paper, along with the results in [15], can be utilized to determine a region in the quadrant of parameters $\beta, \mu > 0$ where the critical curve is possibly located. Moreover, the FK representation allows to compute lower and upper bounds for the infinite-volume free energy.

The paper is organized as follows. In Sections 2.1-2.2 we review the main features of Lorentzian CDTs and describe the Ising model coupled to CDT. In addition, we define the free energy and provide some important properties of the model. In Section 2.3 we present the main results (Theorems 1 and 2). In Section 3.1 we review the FK representation of Ising model coupled to CDT. Sections 3.3 and 3.4 contain the proofs of the theorems formulated in Section 2.3.

2 The notation and results

We start with an introduction to the basic features of CDTs. More details and properties about causal triangulations can be found in [20], [23], [24], [5], [10], [15], [21].

2.1 Two-dimensional Lorentzian models

As in [15], we will work with rooted causal dynamic triangulations of the cylinder $C_N = \mathcal{S} \times [0, N]$, $N = 1, 2, \dots$, which have N bonds (strips)

$\mathcal{S} \times [j, j+1]$. Here \mathcal{S} stands for a unit circle (see Figure 1). Formally, a triangulation $\underline{\mathbf{t}}$ of C_N is called a *rooted causal dynamic triangulation* (CTD) if the following conditions hold:

- each triangular face of $\underline{\mathbf{t}}$ belongs to some strip $\mathcal{S} \times [j, j+1]$, $j = 1, \dots, N-1$, and has all vertices and exactly one edge on the boundary $(\mathcal{S} \times \{j\}) \cup (\mathcal{S} \times \{j+1\})$ of the strip $\mathcal{S} \times [j, j+1]$;
- the number of edges on $\mathcal{S} \times \{j\}$ should be finite for any $j = 0, 1, \dots, N-1$: let $n^j = n^j(\underline{\mathbf{t}})$ be the number of edges on $\mathcal{S} \times \{j\}$, then $1 \leq n^j < \infty$ for all $j = 0, 1, \dots, N-1$.

and have a root face, with the anti-clockwise ordering of its vertices (x, y, z) , where x and y lie in $\mathcal{S} \times \{0\}$.

The CDTs arise naturally when physicists attempt to define a fundamental path integral in quantum gravity. See [6] for a review of the relevant literature; for a rigorous mathematical background of the model, cf. [21]. Additional properties of CDTs have been studied in [23].

A rooted CDT $\underline{\mathbf{t}}$ of C_N is identified with a compatible sequence

$$\underline{\mathbf{t}} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N-1)),$$

where $\mathbf{t}(i)$ is a triangulation of the strip $\mathcal{S} \times [i, i+1]$. The compatibility means that

$$n_{up}(\mathbf{t}(i+1)) = n_{do}(\mathbf{t}(i)), \quad i = 0, \dots, N-2. \quad (2.1)$$

Note that for any edge lying on the slice $\mathcal{S} \times \{i\}$ belongs to exactly two triangles: one up-triangle from $\mathbf{t}(i)$ and one down-triangle from $\mathbf{t}(i-1)$. This provides the following relation: the number of triangles in the triangulation $\underline{\mathbf{t}}$, denoted by $n(\underline{\mathbf{t}})$, is twice the total number of edges on the slices. More precisely, remind that n^i is the number of edges on slice $\mathcal{S} \times \{i\}$. Then, for any $i = 0, 1, \dots, N-1$,

$$n(\underline{\mathbf{t}}) = n_{up}(\mathbf{t}(i)) + n_{do}(\mathbf{t}(i)) = n^i + n^{i+1}. \quad (2.2)$$

In the physical approach to statistical models, the computation of the partition function is the first step towards a deep understanding of the model, enabling for instance the computation of the free energy. Follow this approach, the computation of the partition function for the case of pure CDTs, was first introduced and computed in [7] (see also [21] for a mathematically rigorous account).

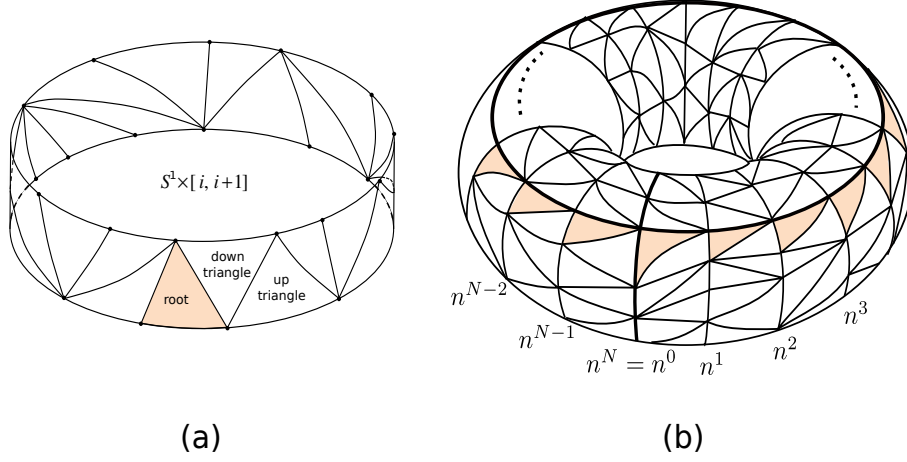


Figure 1: (a) A strip triangulation of $\mathcal{S} \times [j, j+1]$. (b) Geometric representation of a CDT with periodic spatial boundary condition.

Considering triangulations with periodical spatial boundary conditions, i.e. the strip $\mathbf{t}(N-1)$ is compatible with $\mathbf{t}(0)$, and let \mathbb{LT}_N denote the set of causal triangulations on the cylinder C_N with this boundary condition, thus the partition function for rooted CDTs in the cylinder C_N with periodical spatial boundary conditions and for the value of the cosmological constant μ is given by

$$Z_N(\mu) = \sum_{\underline{\mathbf{t}}} e^{-\mu n(\underline{\mathbf{t}})} = \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp \left\{ -\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\}. \quad (2.3)$$

Moreover, the periodical spatial boundary condition on the CDTs permits to write the partition function $Z_N(\mu)$ in a trace-related form

$$Z_N(\mu) = \text{tr} (U^N). \quad (2.4)$$

This gives rise to a transfer matrix $U = \{u(n, n')\}_{n, n'=1, 2, \dots}$ describing the transition from one spatial strip to the next one. It is an infinite matrix with positive entries

$$u(n, n') = \binom{n+n'-1}{n-1} e^{-\mu(n+n')}. \quad (2.5)$$

Employing the N -strip partition function for pure CDTs with periodical boundary condition, defined by the formula (2.3), we define the N -strip

Gibbs probability distribution for pure CDTs

$$\mathbb{Q}_{N,\mu}(\mathbf{t}) = \frac{1}{Z_N(\mu)} e^{-\mu n(\mathbf{t})}. \quad (2.6)$$

The asymptotic properties of the partition function of the model and existence of the weak limit of the measure $\mathbb{Q}_{N,\mu}$ for $\mu \geq \ln 2$ are provided in [21]. These results were obtained by utilizing transfer-matrix formalism and tree parametrization of Lorentzian triangulations. In addition, the transfer-matrix formalism and physical considerations suggest that, as $N \rightarrow \infty$, the partition function is controlled by the largest eigenvalue Λ of the transfer matrix (2.5):

$$Z_N(\mu) = \text{tr } U^N \sim \Lambda^N, \quad (2.7)$$

where

$$\Lambda := \Lambda(\mu) = \left[\frac{1 - \sqrt{1 - 4 \exp(-2\mu)}}{2 \exp(-\mu)} \right]^2. \quad (2.8)$$

That heuristic result was proved in [15].

The following properties hold and will be utilized to prove the main results.

Property 1. (Theorem 1 in [21]). *For any $\mu \geq \ln 2$ the following relation holds:*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \ln Z_N(\mu) = \ln \Lambda(\mu). \quad (2.9)$$

Furthermore, the N -strip Gibbs measure $\mathbb{Q}_{N,\mu}$ converges weakly to a limiting measure \mathbb{Q}_μ .

Property 2. (Proposition 5, [21]). *For any $\mu < \ln 2$, the N -strip partition function $Z_N(\mu)$ exists if*

$$\mu > \ln \left(2 \cos \frac{\pi}{N+1} \right). \quad (2.10)$$

Another proof of Property 1 can be found in [15]. In order to prove this property the authors utilize the transfer-matrix formalism and Krein-Rutman theorem. Inequality (2.10) in Property 2 implies that if $\mu < \ln 2$, then there exists $N_0 \in \mathbb{N}$ such that $Z_N(\mu) = \infty$ if $N > N_0$.

2.2 Ising model on random causal triangulations: Definition and preliminary results

Let $\Delta(\mathbf{t})$ and $\Delta(\mathbf{t}(i))$ denote the set of triangles of the triangulation \mathbf{t} and the set of triangles of the strip $\mathbf{t}(i)$, respectively. As in [15], each triangle

from the triangulation \mathbf{t} is associated with a spin taking values ± 1 . An N -strip configuration of spins is represented by a collection

$$\underline{\sigma} = (\sigma(0), \sigma(1), \dots, \sigma(N-1))$$

where $\sigma(i) \equiv \sigma(\mathbf{t}(i)) := \{\sigma(t)\}_{t \in \Delta(\mathbf{t}(i))}$ is a configuration of spins $\sigma(t)$ over triangles t forming a triangulation $\mathbf{t}(i)$.

We consider a usual ferromagnetic Ising-type energy where two spins $\sigma(t)$ and $\sigma(t')$ interact if their corresponding supporting triangles t, t' share a common edge. In this case the triangles t, t' are referred to as nearest neighbors and denoted by $\langle t, t' \rangle$.

Thus, the Hamiltonian utilized for the annealed model is given by

$$\mathbf{h}(\underline{\sigma}, \mathbf{t}) = - \sum_{\langle t, t' \rangle} \sigma(t) \sigma(t'). \quad (2.11)$$

The partition function for the N -strip Ising model coupled to CDT, at the inverse temperature $\beta > 0$ and the cosmological constant μ , is given by

$$\begin{aligned} \Xi_N(\beta, \mu) = & \sum_{(\mathbf{t}(0), \dots, \mathbf{t}(N-1))} \exp \left\{ -\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\} \\ & \times \sum_{(\sigma(0), \dots, \sigma(N-1))} \prod_{i=0}^{N-1} \exp \left\{ -\beta h(\sigma(i)) - \beta v(\sigma(i), \sigma(i+1)) \right\}, \end{aligned} \quad (2.12)$$

where $n(\mathbf{t}(i))$ stands for the number of triangles in the triangulation $\mathbf{t}(i)$, $h(\sigma(i))$ denotes the inner energy of the configuration $\sigma(i)$, and $v(\sigma(i), \sigma(i+1))$ is the energy of interaction between the neighboring triangles belonging to the adjacent strips $\mathcal{S} \times [i, i+1]$ and $\mathcal{S} \times [i+1, i+2]$.

As in the case of pure CDTs, application of the transfer-matrix formalism to Ising coupled to CDTs suggests that the partition function would be rewritten as

$$\Xi_N(\beta, \mu) = \text{tr } \mathbf{K}^N. \quad (2.13)$$

The formal definition of the operator \mathbf{K} and a proof of the identity (2.13) can be found in [15].

Following [15], we define the operator \mathbf{K} on the Hilbert space $\ell_{\text{T-C}}^2$ where the subscript T-C refers to triangulations and spin-configurations of square-summable functions

$$(\mathbf{t}, \sigma) \mapsto \phi(\mathbf{t}, \sigma), \text{ with } \sum_{\mathbf{t}, \sigma} |\phi(\mathbf{t}, \sigma)|^2 < \infty,$$

where the argument $(\mathbf{t}, \boldsymbol{\sigma})$ runs over single-strip triangulations and supported configurations of spins, with the scalar product $\langle \phi', \phi'' \rangle_{\text{T-C}} = \sum_{\mathbf{t}, \boldsymbol{\sigma}} \phi'(\mathbf{t}, \boldsymbol{\sigma}) \overline{\phi''}(\mathbf{t}, \boldsymbol{\sigma})$ and the induced norm $\|\phi\|_{\text{T-C}}$.

As in the case of the pure CDT, we introduce the N -strip Gibbs probability distribution associated with (2.12) as

$$\mathbb{P}_N^{\beta, \mu}(\mathbf{t}, \boldsymbol{\sigma}) = \frac{1}{\Xi_N(\beta, \mu)} e^{-\mu n(\mathbf{t}) - \beta \mathbf{h}(\boldsymbol{\sigma}, \mathbf{t})}. \quad (2.14)$$

Let $\mathcal{G}_{\beta, \mu}$ be the set of *Gibbs measures* given by the closed convex hull of the set of weak limits:

$$\mathbb{P}^{\beta, \mu} = \lim_{N \rightarrow \infty} \mathbb{P}_N^{\beta, \mu}. \quad (2.15)$$

We define the domain of parameters where the weak limit Gibbs distribution exists, as follows

$$\Gamma = \{(\beta, \mu) \in \mathbb{R}_+^2 : \mathcal{G}_{\beta, \mu} \neq \emptyset\}. \quad (2.16)$$

Further, the critical curve γ_{cr} for the Ising model coupled to CDT is defined as

$$\gamma_{cr} = \partial\Gamma \cap \mathbb{R}_+^2. \quad (2.17)$$

This critical curve is of major importance, since it splits the parameter space into two following regions: a region where the Gibbs distributions exist, and a region where the Gibbs distribution does not exist, due to large Boltzman factors associated with the triangulations which results in an infinite partition function. This behaviour is observed at zero and infinite temperature in the annealed model.

In order to study behaviour of the critical curve we focus on the thermodynamic function which is defined as

$$\phi(\beta, \mu) = \lim_{N \rightarrow \infty} \phi_N(\beta, \mu) = \frac{1}{N} \ln \Xi_N(\beta, \mu). \quad (2.18)$$

Note that,

$$\begin{aligned} \phi_N(0^+, \mu) = \lim_{\beta \rightarrow 0^+} \phi_N(\beta, \mu) &= \frac{1}{N} \ln \left(\sum_{\mathbf{t}} \exp \left\{ -\mu \sum_{i=0}^{N-1} n(\mathbf{t}(i)) \right\} \sum_{\boldsymbol{\sigma}} 1 \right) \\ &= \frac{1}{N} \ln Z_N(\mu - \ln 2). \end{aligned}$$

Utilizing (2.9), it is easy to show that $\phi_N(0^+, \mu)$ converges to $\ln \Lambda(\mu - \ln 2)$ as $N \rightarrow \infty$. This result implies that the free energy exists at infinitely high temperature for $\mu > 2 \ln 2$,

$$\phi(0^+, \mu) = \lim_{N \rightarrow \infty} \phi_N(0^+, \mu) = \ln \Lambda(\mu - \ln 2).$$

It can be noticed that, if $\beta = 0$, and $\mu \rightarrow 2 \ln 2$, then $\phi(0^+, \mu) \rightarrow 0$. Thus, utilizing the results (2.9) and (2.10) (see [21] for details), we find the infinite-volume free energy $\phi(0^+, \mu)$ at infinitely high temperature

$$\phi(0^+, \mu) = \begin{cases} \ln \Lambda(\mu - \ln 2) < +\infty & , \quad \mu(0^+) > 2 \ln 2 \\ 0 & , \quad \mu(0^+) = 2 \ln 2 \\ +\infty & , \quad \mu(0^+) < 2 \ln 2 \end{cases} \quad (2.19)$$

In addition, if the temperature tends to infinity, we obtain that the annealed model is a pure CDT model with a critical parameter $\mu_{cr}(0^+) = 2 \ln 2$. Furthermore, for all $\mu \geq 2 \ln 2$ the Gibbs measure is unique, and as in the case of pure CDTs there exist two types of triangulations with respect to this measure. If $\mu > 2 \ln 2$ we obtain subcritical triangulations. Utilizing the identity (2.19) and the results obtained in [11] and [21] it can be showed that in this case the triangulations have a bounded diameter with probability 1, which is essentially one dimensional random geometry. The boundary value $\mu = 2 \ln 2$ yields a two dimensional random geometry with Hausdorff dimension $d_H = 2$ (see [10], [11] and [24] for details).

In this paper we show that for any $0 < \beta < \infty$ the infinite-volume free energy of the annealed model has the same behaviour as (2.19) except for a small interval $I(\beta) = [f_1(\beta), f_2(\beta)]$ where the behaviour of the function is unknown. On the other hand, according to the numerical results (see [3], [4], [8]) there exists a unique value $\tilde{\mu}_{cr} = \tilde{\mu}_{cr}(\beta) \in I(\beta)$ such that $\phi(\beta, \mu) = +\infty$ for $\mu < \tilde{\mu}_{cr}(\beta)$, $\phi(\beta, \mu) = 0$ for $\mu = \tilde{\mu}_{cr}(\beta)$, and $\phi(\beta, \mu) < +\infty$ for $\mu > \tilde{\mu}_{cr}(\beta)$, however the exact value of $\tilde{\mu}(\beta)$ has not been computed yet. The interval $I(\beta)$ is computed explicitly in the next section.

Note that, if $0 < \beta < \infty$, for any triangulation \mathbf{t} the energy of any spin configuration σ on \mathbf{t} is larger than or equal to the energy of a pure configuration (all +’s or all -’s). i.e., $\mathbf{h}(\sigma, \mathbf{t}) \geq -3/2n(\mathbf{t})$. Thus, for any $\beta > 0$ the following inequality holds

$$\phi_N(\beta, \mu) < \frac{1}{N} \ln \left(\sum_{\mathbf{t}} e^{-(\mu - \frac{3}{2}\beta - \ln 2)n(\mathbf{t})} \right) = \frac{1}{N} \ln Z_N \left(\mu - \frac{3}{2}\beta - \ln 2 \right).$$

Therefore,

$$\phi(\beta, \mu) = \lim_{N \rightarrow \infty} \phi_N(\beta, \mu) \leq \ln \Lambda \left(\mu - \frac{3}{2}\beta - \ln 2 \right). \quad (2.20)$$

Hence, the inequality $\mu > \frac{3}{2}\beta + 2 \ln 2$ provides a sufficient condition for subcritical behaviour (uniqueness of Gibbs measure) of the Ising model coupled

to CDTs, and provides a region where the infinite-volume free energy exists and is finite, i.e., does not have singularities.

Now, based on (2.9) and (2.20) an upper bound for the critical curve of the annealed model is given by

$$\mu < \frac{3}{2}\beta + 2 \ln 2 \text{ for all } (\beta, \mu) \in \gamma_{cr}. \quad (2.21)$$

Obviously, the upper bound given by (2.21) is not sharp for all β . For example, for the low values of β , which correspond to high temperature, the upper bounds provided by [15] and [3] are more accurate. In this paper we focus on finding lower and upper bounds for the critical curve for all values of β .

2.3 The main results

In the previous subsection we examine basic notions of statistical mechanics of the model and review some properties about the phase space, the free energy and Gibbs distributions. In this section we focus on presenting the main results of this paper. Utilizing these results we provide a first phase diagram for the Ising Model coupled to CDTs in the parameters β and μ , and compute an approximation of the infinite-volume free energy. The exact phase diagram at infinite temperature was computed in the previous subsection.

In order to construct an upper bound for the critical curve, we define the function $\lambda(\beta, \mu)$ as follows

$$\lambda(\beta, \mu) = c^2 (m^2 + 1) (\cosh 2\beta) \left(1 + \sqrt{1 - \frac{1}{(\cosh 2\beta)^2} \frac{(m^2 - 1)^2}{(m^2 + 1)^2}} \right), \quad (2.22)$$

where c and m are determined by

$$\begin{aligned} c &= \frac{\exp(\beta - \mu)}{e^{2\beta}(1 - \exp(\beta - \mu))^2 - e^{-2\mu}}, \\ m &= e^{2\beta} + (1 - e^{4\beta}) \exp(-(\beta + \mu)). \end{aligned}$$

Let $\psi(\beta)$ be a strictly increasing function defined as

$$\psi(\beta) = \inf\{\mu \in \mathbb{R}^+ : \lambda(\beta, \mu) < 1\}, \quad \text{for } \beta > 0. \quad (2.23)$$

It is easy to see that

$$\psi(0) = \lim_{\beta \rightarrow 0^+} \psi(\beta) = 2 \ln 2. \quad (2.24)$$

In addition, it is well known that in the region $\{(\beta, \mu) : \mu > \psi(\beta)\}$ the free energy $\phi(\beta, \mu)$ is finite. Consequently, in this region the annealed model has a unique Gibbs measure (see [15] for details).

Let $\mathbf{t}_1, \dots, \mathbf{t}_k$ be triangulations of a single strip $\mathcal{S} \times [0, 1]$ and $\boldsymbol{\sigma}_1, \dots, \boldsymbol{\sigma}_k$ be their corresponding spin configurations. Given $0 \leq i_1 < \dots < i_k \leq N-1$, we define a finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k} = \mathcal{C}_{i_1, \dots, i_k}^{(\mathbf{t}_1, \boldsymbol{\sigma}_1), \dots, (\mathbf{t}_k, \boldsymbol{\sigma}_k)}$ as follows

$$\mathcal{C}_{i_1, \dots, i_k} = \{(\underline{\mathbf{t}}, \underline{\boldsymbol{\sigma}}) : (\mathbf{t}(i_1), \boldsymbol{\sigma}(i_1)) = (\mathbf{t}_1, \boldsymbol{\sigma}_1), \dots, (\mathbf{t}(i_k), \boldsymbol{\sigma}(i_k)) = (\mathbf{t}_k, \boldsymbol{\sigma}_k)\}. \quad (2.25)$$

Theorem 1. *If $(\beta, \mu) \in \mathbb{R}_+^2$, where*

$$\mu < \max \left\{ 2 \ln 2, \frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \right\},$$

then there exist $N_0 \in \mathbb{N}$ such that if $N > N_0$ the partition function $\Xi_N(\beta, \mu) = \infty$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by utilizing the standard formula with a normalising denominator $\Xi_N(\beta, \mu)$, consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$.

Formally Theorem 1 states that, for any finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k}$ we obtain $\mathbb{P}_N^{\beta, \mu}(\mathcal{C}_{i_1, \dots, i_k}) = 0$ for $N > N_0 \geq \max\{i_1, \dots, i_k\}$.

Let β_1^*, β_2^* be positive solutions of the following equations

$$2 \ln 2 = \frac{3}{2} \ln(2 \sinh \beta) + \ln 2, \quad (2.26)$$

$$\frac{3}{2} \beta + 2 \ln 2 = \psi(\beta), \quad (2.27)$$

respectively. Theorem 1, along with the results in [15], provides two-side bounds for the critical curve.

Theorem 2. *The critical curve γ_{cr} satisfies the following inequalities.*

(i) *If $(\beta, \mu) \in \gamma_{cr}$ and $0 < \beta < \beta_1^*$, then*

$$2 \ln 2 \leq \mu < \psi(\beta).$$

(ii) *If $(\beta, \mu) \in \gamma_{cr}$ and $\beta_1^* \leq \beta < \beta_2^*$, then*

$$\frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \leq \mu < \psi(\beta).$$

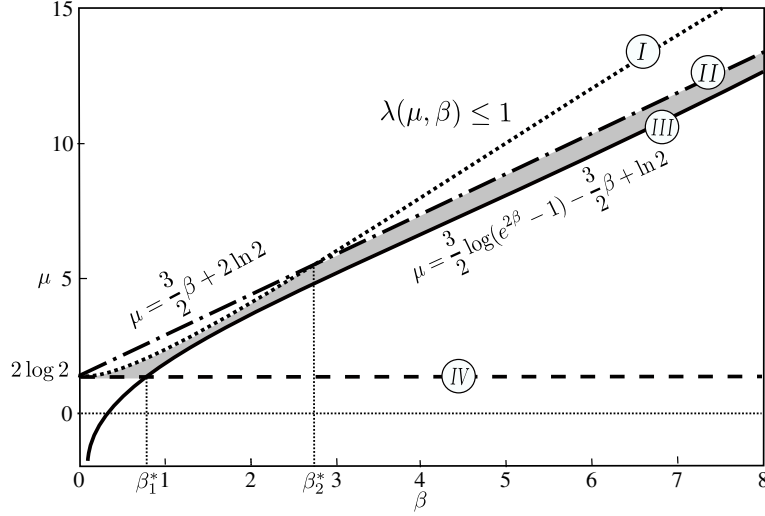


Figure 2: The area above the minimum of the dotted curve I (graph of the function ψ defined in (2.23)) and dash-dotted line II is where the limiting Gibbs probability measure exists and is unique. The critical curve lies in the region below the dotted curve I and dash-dotted line II but above the continuous curve III and dashed line IV.

(iii) If $(\beta, \mu) \in \gamma_{cr}$ and $\beta_2^* \leq \beta < \infty$, then

$$\frac{3}{2} \ln(2 \sinh \beta) + \ln 2 \leq \mu < \frac{3}{2} \beta + 2 \ln 2.$$

Remark 1. Utilizing (2.24) and (i) of Theorem 2, we obtain the exact value of the critical curve at infinite temperature, i.e., $(0, 2 \ln 2) \in \gamma_{cr}$. Employing the results of [21] we conclude that the set of Gibbs measures at infinite temperature is a single point for $\mu \geq 2 \ln 2$, and is empty for $\mu < 2 \ln 2$.

As a by-product of the proof of Theorems 1 and 2, we obtain a lower and upper bound for the infinite-volume free energy. These bounds are provided in the following corollary.

Let $I(\beta)$ denote the interval $[f_1(\beta), f_2(\beta)]$, where

$$f_1(\beta) = \max \left\{ \ln 2, \frac{3}{2} \ln(2 \sinh \beta) \right\}, \quad (2.28)$$

$$f_2(\beta) = \min \left\{ \psi(\beta) - \ln 2, \frac{3}{2} \beta + \ln 2 \right\}, \quad (2.29)$$

Corollary 1. For (β, μ) such that $\mu > f_2(\beta)$, the free energy $\phi(\beta, \mu)$ for the Ising model coupled to CDTs is finite and satisfies the following inequalities,

$$\ln \Lambda(\mu - f_1(\beta)) \leq \phi(\beta, \mu) \leq \ln \Lambda(\mu - f_2(\beta)).$$

Here $\Lambda(s)$ is given by (2.8).

Finally, we obtain the behaviour of the free energy for the annealed model.

Corollary 2. For $\beta > 0$ fixed, the free energy $-\phi(\beta, \mu)$ is an increasing function in μ , for all $\mu > f_2(\beta) + \ln 2$ and $\phi(\beta, \mu) = \infty$ for $\mu < f_1(\beta) + \ln 2$. In addition, $\lim_{\beta \rightarrow 0^+} I(\beta) = \{\ln 2\}$ and $\phi(0^+, \mu) = \ln \Lambda(\mu - \ln 2)$.

Remember that the exact solution of the free energy at $\beta = 0$ was computed in (2.19). The result presented in Corollary 2 is illustrated in Figure 3.

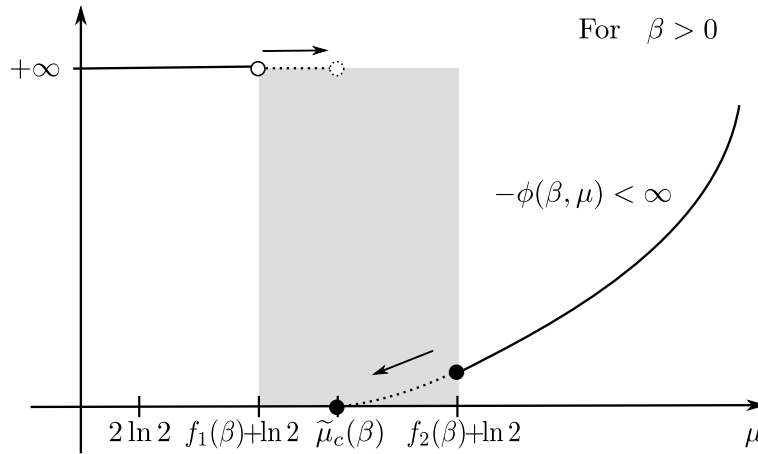


Figure 3: Behaviour of the free energy $\phi(\beta, \mu)$ for β fixed. In the gray region we show a possible behaviour of analytic continuation of the free energy on the right and left side, according to the numerical results given in [3], [4], [8].

3 Proof of Theorem 1 and 2

We start this section with a brief review of the existing results for representation of the Ising model in terms of a two continuum random-cluster model. We apply the results obtained in [2] and [16] to Lorentzian triangulations with periodical boundary condition.

3.1 FK representation for Ising model coupled to CDT

As mentioned above, each triangle from the triangulation $\underline{\mathbf{t}}$ is associated with a spin taking values ± 1 . This is equivalent to associating spins on each vertex of the dual Lorentzian triangulation. Starting from this section we shall use the same notation $\underline{\mathbf{t}}$ for the dual Lorentzian triangulation with a periodical boundary condition. Note that the Ising model defined in Subsection 2.2 was defined on dual triangulation.

Let $\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}$ be the partition function of the Ising model on a fixed dual Lorentzian triangulation $\underline{\mathbf{t}}$, at inverse temperature $\beta > 0$

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} = \sum_{\underline{\sigma} \in \Omega_{\underline{\mathbf{t}}}} \exp\{-\beta \mathbf{h}(\underline{\sigma}, \underline{\mathbf{t}})\}. \quad (3.1)$$

Here $\mathbf{h}(\underline{\sigma}, \underline{\mathbf{t}})$ represents the energy of the configuration $\underline{\sigma} \in \Omega_{\underline{\mathbf{t}}}$, defined by (2.11). Thus, utilizing (3.1) the partition function for the N -strip Ising model coupled to CDT, at the inverse temperature $\beta > 0$ for the cosmological constant μ , can be rewritten as

$$\Xi_N(\beta, \mu) = \sum_{\underline{\mathbf{t}}} e^{-\mu n(\underline{\mathbf{t}})} \mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}. \quad (3.2)$$

For any dual Lorentzian triangulation with periodical boundary condition $\underline{\mathbf{t}}$ and any edge $e = \langle i, j \rangle$ of $\underline{\mathbf{t}}$, we select a Poisson process ξ_e of points in $\{e\} \times \mathbb{R}$ with intensity 2, such that the members of the family of Poisson processes $\{\xi_e : e \in E(\underline{\mathbf{t}})\}$ are independent of each other, where $E(\underline{\mathbf{t}})$ denotes the set of edges of the dual Lorentzian triangulation $\underline{\mathbf{t}}$. Note that $|E(\underline{\mathbf{t}})| = \frac{3}{2}n(\underline{\mathbf{t}})$ for all $\underline{\mathbf{t}} \in \mathbb{LT}_N$.

Let $\mathbb{P}_{\beta, \underline{\mathbf{t}}}$ denote the probability associated with the family of Poisson processes on the interval $[0, \beta]$. An arrival of $\xi_{e=\langle i, j \rangle}$ at time t can be geometrically interpreted as a link between neighbor vertices i, j at time t . We say that two vertices t, t' of $\underline{\mathbf{t}}$ (not necessarily neighbor vertices) are connected, if and only if there exists a path, in the sense of continuum percolation, connecting t and t' (see [14] for an overview). This property will be denoted by $t \leftrightarrow t'$. The relation \leftrightarrow generates a partition of the set of vertices into clusters. Given a realization ξ of the Poisson processes $\{\xi_e : e \in E(\underline{\mathbf{t}})\}$ on $[0, \beta]$, we denote by $k(\xi)$ the number of clusters in the realization ξ .

Papers [1], [2] and [16] derived a representation of the Ising model in terms of a continuum random-cluster model. Application of this representation to an Ising model without external magnetic field on a fixed dual Lorentzian triangulation $\underline{\mathbf{t}}$, yields the following representation for the partition function $\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}$.

Proposition 1. *Let $\underline{\mathbf{t}} \in \mathbb{LT}_N$ and $\beta > 0$. Then the following identity holds*

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} = e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} \int 2^{k(\xi)} \mathbb{P}_{\beta, \underline{\mathbf{t}}}(d\xi). \quad (3.3)$$

Utilizing the N -strip Gibbs probability distribution $\mathbb{Q}_{N, \mu}$, defined in (2.6) for the pure CDT case, (3.2) and Proposition 1 we obtain the FK representation of the partition function for the annealed model, at the inverse temperature $\beta > 0$ and for the cosmological constant μ

$$\Xi_N(\beta, \mu) = Z_N(r) \sum_{\underline{\mathbf{t}} \in \mathbb{LT}_N} \left\{ \int 2^{k(\xi)} \mathbb{P}_{\beta, \underline{\mathbf{t}}}(d\xi) \right\} \mathbb{Q}_{N, r}(\underline{\mathbf{t}}), \quad (3.4)$$

where $r = \mu - \frac{3}{2}\beta$ and $Z_N(\cdot)$ is defined by (2.3). The result (3.4) provides asymptotic behaviour for the free energy, for large β , as follows.

Let $\underline{\mathbf{t}} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N-1))$ be a Lorentzian triangulation. If β is large enough then the probability that all the Poisson processes ξ_e , $e \in E(\underline{\mathbf{t}})$ had at least one arrival in the time interval $[0, \beta]$ tends to 1, which implies that almost all vertices are linked, that is, $\mathbb{P}_{\beta, \underline{\mathbf{t}}}(k(\xi) = 1) \approx 1$. Thus, utilizing the representation (3.4) we obtain that

$$\Xi_N(\beta, \mu) \approx 2Z_N\left(\mu - \frac{3}{2}\beta\right)$$

Consequently, the critical inequality

$$\mu > \frac{3}{2}\beta + \ln 2$$

gives a necessary (and probability tight) criticality condition for the annealed model. Therefore, for any β , which is sufficiently large, the free the free energy have the following behaviour,

$$\phi(\beta, \mu) \approx \ln \Lambda \left(\mu - \frac{3}{2}\beta \right).$$

Furthermore, this asymptotic property establish that for large β

$$d \left(\gamma_{cr}, \left\{ \left(\beta, \frac{3}{2}\beta + \ln 2 \right) : \beta \text{ large enough} \right\} \right) \approx 0,$$

where $d(\cdot, \cdot)$ is the Euclidean distance between two sets. A similar heuristic analysis was presented and confirmed by numerical simulations in [3].

3.2 Trivial lower bound for the critical curve

The purpose of this section is to study the behaviour of the partition function Ξ_N as a function of β , for $\beta \geq 0$, $\mu > 2 \ln 2$, $N \in \mathbb{N}$. We further utilize the properties of this function in order to obtain a lower bound (not sharp) for the critical curve.

Proposition 2. *If $\beta_1 > \beta_2 > 0$, then*

$$\mathcal{Z}_N^{\beta_1, \mathbf{t}} \geq \mathcal{Z}_N^{\beta_2, \mathbf{t}},$$

for any $\mathbf{t} \in \mathbb{LT}_N$.

Proof. Utilizing the definition of the partition function $\mathcal{Z}_N^{\beta, \mathbf{t}}$, it is easy to obtain the following expression

$$\frac{\partial \mathcal{Z}_N^{\beta, \mathbf{t}}}{\partial \beta} = \sum_{\underline{\sigma}} \sum_{\langle t, t' \rangle} \sigma(t) \sigma(t') e^{-\beta \mathbf{h}(\underline{\sigma}, \mathbf{t})} = \mathcal{Z}_N^{\beta, \mathbf{t}} \sum_{\underline{\sigma}} \sum_{\langle t, t' \rangle} \sigma(t) \sigma(t') \mu_{\beta}^{\mathbf{t}}(\underline{\sigma}),$$

where $\mu_{\beta}^{\mathbf{t}}$ denotes the Gibbs measure of an Ising model on \mathbf{t} , given by

$$\mu_{\beta}^{\mathbf{t}}(\underline{\sigma}) = \frac{1}{\mathcal{Z}_N^{\beta, \mathbf{t}}} \exp\{-\beta \mathbf{h}(\underline{\sigma}, \mathbf{t})\},$$

where $\mathbf{h}(\underline{\sigma}, \mathbf{t})$ is defined by (2.11). Thus, implementation of the first Griffiths inequality (see [25] and [26] for an overview) yields

$$\frac{\partial \mathcal{Z}_N^{\beta, \mathbf{t}}}{\partial \beta} = \mathcal{Z}_N^{\beta, \mathbf{t}} \sum_{\langle t, t' \rangle} \int \sigma_{\{t, t'\}} \mu_{\beta}^{\mathbf{t}}(\underline{\sigma}) \geq 0.$$

This completes the proof. \square

Now, in particular, $\mathcal{Z}_N^{\beta, \mathbf{t}} \geq \mathcal{Z}_N^{0, \mathbf{t}} = e^{n(\mathbf{t}) \ln 2}$ for all $\beta > 0$. Thus, the following inequality can be derived

$$\Xi_N(\beta, \mu) \geq Z_N(\mu - \ln 2). \quad (3.5)$$

Utilizing property (2.10) and (3.5), we obtain the trivial lower bound for the critical curve,

$$\text{if } (\beta, \mu) \in \gamma_{cr}, \text{ then } \mu > 2 \ln 2, \quad (3.6)$$

(see Figure 2). In addition, it follows from (3.5) that the free energy satisfies the inequality $\phi(\beta, \mu) \geq \ln \Lambda(\mu - \ln 2)$, for $\beta \geq 0$ and $\mu \geq 2 \ln 2$. Finally, this lower bound along with the results in [15] prove that $(0, 2 \ln 2) \in \gamma_{cr}$, and that $\phi(0^+, \mu) = \ln \Lambda(\mu - \ln 2)$ for all $\mu \geq 2 \ln 2$.

3.3 Proof of Theorem 1

Let $\underline{\mathbf{t}}$ be a dual Lorentzian triangulation on the cylinder C_N . Given $1 \leq k \leq n(\underline{\mathbf{t}})$, we define the set of realizations ξ which splits the set of vertices in k clusters,

$$\mathbf{\Pi}_k = \{\text{all realization } \xi \text{ of } \{\xi_{\langle t, t' \rangle}\} \text{ such that } k(\xi) = k\}. \quad (3.7)$$

Thus, we obtain the following representation of (3.3)

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} = e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} \sum_{k=1}^{n(\underline{\mathbf{t}})} 2^k \mathbb{P}_{\beta, \underline{\mathbf{t}}}(\mathbf{\Pi}_k). \quad (3.8)$$

Let $\xi \in \mathbf{\Pi}_k$ and let $\{\mathcal{C}_l\}_{l=1}^k$ be the corresponding cluster decomposition of the set $\Delta(\underline{\mathbf{t}})$. In fact, each cluster \mathcal{C}_l is a subgraph of $\underline{\mathbf{t}}$ formed by vertices V_l and edges E_l . Note that \mathcal{C}_l is a random variable and that the cluster decomposition $\{\mathcal{C}_l\}_{l=1}^k$ can include isolated vertices. Denote by $\eta_l = |V_l|$ and $\kappa_l = |E_l|$, the number of vertices (triangles) in cluster \mathcal{C}_l and the number of edges in \mathcal{C}_l , respectively. Note that for any decomposition $\{\mathcal{C}_l\}_{l=1}^k$, κ_l and η_l depend on the geometry of the cluster \mathcal{C}_l . Note also that $\sum_{l=1}^k \eta_l = n(\underline{\mathbf{t}})$.

Now, denote by $\pi(\underline{\mathbf{t}})$ the set of all maximal unordered partitions of $\Delta(\underline{\mathbf{t}})$, i.e. an element of $\pi(\underline{\mathbf{t}})$ is an unordered n -ple $\{\mathcal{C}_1 = (V_1, E_1), \dots, \mathcal{C}_n = (V_n, E_n)\}$ of maximal connected subgraphs \mathcal{C}_i , with $1 \leq n \leq n(\underline{\mathbf{t}})$, such that for any $i, j \in I_n = \{1, 2, \dots, n\}$, $V_i \subset \Delta(\underline{\mathbf{t}})$, $V_i \neq \emptyset$, $V_i \cap V_j = \emptyset$ and $\cup V_i = \Delta(\underline{\mathbf{t}})$. A graph \mathcal{C}_i is maximal in the sense that: if t, t' are nearest neighbor vertices in \mathcal{C}_i , then $\{t, t'\} \in E_i$. Given a subgraph \mathcal{C} belonging to some element of $\pi(\underline{\mathbf{t}})$, we define the set of spanning subgraphs of \mathcal{C} as

$$\text{Span}(\mathcal{C}) = \{\gamma : \gamma \text{ is a connected subgraph of } \mathcal{C} \text{ with } |V(\gamma)| = |V(\mathcal{C})|\},$$

that is, if $\gamma \in \mathcal{C}$, then γ and \mathcal{C} have exactly the same vertex set.

Finally, the probability that two nearest neighbor vertices t, t' are linked is given by $\mathbb{P}_{\beta, \underline{\mathbf{t}}}(t \leftrightarrow t') = 1 - e^{-2\beta}$. Then, denoting by $p = \mathbb{P}_{\beta, \underline{\mathbf{t}}}(t \leftrightarrow t')$, we can rewrite the partition function $\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}$ as follows

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} = e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} (1-p)^{\frac{3}{2}n(\underline{\mathbf{t}})} \sum_{k=1}^{n(\underline{\mathbf{t}})} 2^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})} \rho(\mathcal{C}_1) \cdots \rho(\mathcal{C}_k), \quad (3.9)$$

where

$$\rho(\mathcal{C}) = \sum_{\gamma \in \text{Span}(\mathcal{C})} \left(\frac{p}{1-p} \right)^{|E(\gamma)|}.$$

Here $|E(\gamma)|$ denotes the number of edges in the subgraph γ . Notice that $|E(\gamma)| \geq 0$ for all $\gamma \in \mathcal{C}$.

A convenient expansion parameter for our analysis is $u = \frac{p}{1-p} \in [0, \infty)$. We now are interested in deriving expansions of the function $\rho(\mathcal{C})$,

$$\rho(\mathcal{C}) = \sum_{\gamma \in \text{Span}(\mathcal{C})} u^{|E(\gamma)|},$$

for values $u > 1$ and $u < 1$. Note that by the definition of a spanning subgraph, it is easy to see that for any $\gamma \in \text{Span}(\mathcal{C})$ the following inequalities hold

$$|\mathcal{C}| - 1 \leq |E(\gamma)| \leq \frac{3}{2}|\mathcal{C}| - 1, \text{ if } \mathcal{C} \subsetneq \Delta(\underline{\mathbf{t}}). \quad (3.10)$$

If $\mathcal{C} = \underline{\mathbf{t}}$ the unique spanning subgraph γ that do not satisfy the right-hand side of inequality (3.10) is $\gamma = \mathcal{C}$. In this case, the graph γ has $|V(\gamma)| = n(\underline{\mathbf{t}})$ vertices and $|E(\gamma)| = \frac{3}{2}n(\underline{\mathbf{t}})$ edges. If $\gamma \subsetneq \mathcal{C} = \underline{\mathbf{t}}$ the inequality (3.10) is satisfied.

In order to obtain lower bounds for the representation (3.9), we implement inequality (3.10) for two different cases: $u > 1$ and $u < 1$.

The case $u > 1$ (equivalent to $\beta > \frac{\ln 2}{2}$). Utilizing the inequality (3.10), we obtain that the following inequality holds for $\gamma \neq \underline{\mathbf{t}}$

$$u^{|\mathcal{C}|-1} \leq u^{|E(\gamma)|} \leq u^{\frac{3}{2}|\mathcal{C}|-1}.$$

Thus, if $\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})$ and $k \geq 2$ (the case $k = 1$ will treat separately later), we obtain

$$u^{n(\underline{\mathbf{t}})-k} \prod_{i=1}^k f(\mathcal{C}_i) \leq \prod_{i=1}^k \rho(\mathcal{C}_i) \leq u^{\frac{3}{2}n(\underline{\mathbf{t}})-k} \prod_{i=1}^k f(\mathcal{C}_i), \quad (3.11)$$

where $f(\mathcal{C}) = \sum_{\gamma \in \text{Span}(\mathcal{C})} 1$, that is, the number of spanning subgraphs contained in \mathcal{C} . Further, we rewrite the partition function $\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}$ as follows

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} = e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} (1-p)^{\frac{3}{2}n(\underline{\mathbf{t}})} \left(2\rho(\Delta(\underline{\mathbf{t}})) + \sum_{k=2}^{n(\underline{\mathbf{t}})} 2^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})} \rho(\mathcal{C}_1) \cdots \rho(\mathcal{C}_k) \right),$$

where $\rho(\Delta(\underline{\mathbf{t}})) = \sum_{\gamma \in \text{Span}(\Delta(\underline{\mathbf{t}}))} u^{|E(\gamma)|}$.

Now, observing that $e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})}(1-p)^{\frac{3}{2}n(\underline{\mathbf{t}})} = e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})}$ and employing the inequality (3.11), we obtain the following lower bound

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} \geq e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})} \left\{ 2\rho(\Delta(\underline{\mathbf{t}})) + u^{n(\underline{\mathbf{t}})} \sum_{k=2}^{n(\underline{\mathbf{t}})} \left(\frac{2}{u}\right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})} \prod_{i=1}^k f(\mathcal{C}_i) \right\} \quad (3.12)$$

Notice that, given any decomposition $\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})$, $f(\mathcal{C}_i) \geq 1$ for all $i \in \{1, \dots, k\}$, where equality holds only if \mathcal{C}_i is a spanning tree. Further, if the degree of freedom, k is large enough, there exists positive proportion of decompositions $\{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ of $\underline{\mathbf{t}}$ with $f(\mathcal{C}_i) = 1$ for all k . Thus, the lower bound can be obtain as $\prod_{i=1}^k f(\mathcal{C}_i) \geq 1$. For example, for $k = n(\underline{\mathbf{t}}), n(\underline{\mathbf{t}}) - 1$ and $n(\underline{\mathbf{t}}) - 2$, all elements of any decomposition are spanning trees. Substituting the lower bound in (3.12), we get the following inequalities

$$\begin{aligned} \mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} &\geq e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})} \left\{ 2\rho(\Delta(\underline{\mathbf{t}})) + u^{n(\underline{\mathbf{t}})} \sum_{k=2}^{n(\underline{\mathbf{t}})} \left(\frac{2}{u}\right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})} 1 \right\} \\ &\geq e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})} \left\{ 2\rho(\Delta(\underline{\mathbf{t}})) + u^{n(\underline{\mathbf{t}})} \sum_{k=1}^{n(\underline{\mathbf{t}})-1} \left(\frac{2}{u}\right)^{k+1} \binom{n(\underline{\mathbf{t}})-1}{k} \right\} \\ &= e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})} \left\{ 2\rho(\Delta(\underline{\mathbf{t}})) + 2u^{n(\underline{\mathbf{t}})-1} \left(\left(1 + \frac{2}{u}\right)^{n(\underline{\mathbf{t}})-1} - 1 \right) \right\} \\ &= e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})} \left\{ 2\rho(\Delta(\underline{\mathbf{t}})) - 2u^{n(\underline{\mathbf{t}})-1} + 2(2+u)^{n(\underline{\mathbf{t}})-1} \right\} \end{aligned}$$

Now, the function $\rho(\Delta(\underline{\mathbf{t}}))(u)$ can be rewritten as

$$\rho(\Delta(\underline{\mathbf{t}})) = u^{\frac{3}{2}n(\underline{\mathbf{t}})} + \sum_{\substack{\gamma \in \text{Span}(\Delta(\underline{\mathbf{t}})): \\ |E(\gamma)| < \frac{3}{2}n(\underline{\mathbf{t}})}} u^{|E(\gamma)|}.$$

If $\mu > 1$ and number of strip N is large enough, we can conclude that the behaviour of the function $\rho(\Delta(\underline{\mathbf{t}}))(u)$ can be described by the term $u^{\frac{3}{2}n(\underline{\mathbf{t}})}$. In fact, it is easy to see that $\rho(\Delta(\underline{\mathbf{t}})) > u^{\frac{3}{2}n(\underline{\mathbf{t}})}$ for any triangulation $\underline{\mathbf{t}}$. Utilizing this result, we derive the following lower bound for the partition function $\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}$ of the Ising model on any triangulation $\underline{\mathbf{t}}$ and $N \in \mathbb{N}$,

$$\begin{aligned} \mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} &\geq 2e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} p^{\frac{3}{2}n(\underline{\mathbf{t}})} - 2e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} p^{n(\underline{\mathbf{t}})-1} (1-p)^{\frac{1}{2}n(\underline{\mathbf{t}})+1} \\ &\quad + 2e^{\frac{3}{2}\beta n(\underline{\mathbf{t}})} (2-p)^{n(\underline{\mathbf{t}})-1} (1-p)^{\frac{1}{2}n(\underline{\mathbf{t}})+1}. \end{aligned} \quad (3.13)$$

Implementation of identity (3.4), and lower bound (3.13) provides the following lower bound for the partition function of the annealed model,

$$\begin{aligned} \Xi_N(\beta, \mu) \geq & 2Z_N \left(\mu - \frac{3}{2}\beta - \frac{3}{2}\ln p \right) - 2Z_N \left(\mu - \frac{3}{2}\beta - \ln p \sqrt{1-p} \right) + \\ & 2Z_N \left(\mu - \frac{3}{2}\beta - \ln(2-p)\sqrt{1-p} \right). \end{aligned} \quad (3.14)$$

Thus, utilizing the asymptotic property given in (2.10), we obtain that the partition function $\Xi_N(\beta, \mu)$ exists only if

$$\mu > \frac{3}{2}\beta + \ln 2 + \frac{3}{2}\ln \left(1 - e^{-2\beta} \right) + \ln \left(\cos \frac{\pi}{N+1} \right) \quad \text{and} \quad \beta > \frac{\ln 2}{2}.$$

Now, let us discuss the case where $N \rightarrow \infty$.

Proposition 3. *If $(\beta, \mu) \in \mathbb{R}_+^2$ such that*

$$\mu < \frac{3}{2}\beta + \ln 2 + \frac{3}{2}\ln \left(1 - e^{-2\beta} \right) \quad \text{and} \quad \beta > \frac{\ln 2}{2},$$

then there exists $N_0 \in \mathbb{N}$ such that the partition function $\Xi_N(\beta, \mu) = +\infty$ whenever $N > N_0$. Moreover, the Gibbs distribution $\mathbb{P}_N^{\beta, \mu}$ with periodic boundary conditions cannot be defined by the standard formula with $\Xi_N(\beta, \mu)$ being a normalising denominator. Consequently, there is no limiting probability measure $\mathbb{P}^{\beta, \mu}$ as $N \rightarrow \infty$, which implies that for any finite-dimensional cylinder $\mathcal{C}_{i_1, \dots, i_k}$, $\mathbb{P}_N^{\beta, \mu}(\mathcal{C}_{i_1, \dots, i_k}) = 0$ whenever $N > N_0 \geq \max\{i_1, \dots, i_k\}$.

The case $u < 1$ (equivalent to $\beta < \frac{\ln 2}{2}$). Similarly to the first case, utilizing (3.10) we obtain the following inequalities

$$u^{\frac{3}{2}|\mathcal{C}|-1} \leq u^{|E(\gamma)|} \leq u^{|\mathcal{C}|-1},$$

which hold for all spanning subgraphs $\gamma \neq \underline{\mathbf{t}}$. Thus, if $\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})$ and $k \geq 2$ we obtain

$$u^{\frac{3}{2}n(\underline{\mathbf{t}})-k} \prod_{i=1}^k f(\mathcal{C}_i) \leq \prod_{i=1}^k \rho(\mathcal{C}_i) \leq u^{n(\underline{\mathbf{t}})-k} \prod_{i=1}^k f(\mathcal{C}_i). \quad (3.15)$$

Employing the inequality (3.15), we obtain the lower bound for the partition function $\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}}$ on dual triangulation $\underline{\mathbf{t}}$,

$$\mathcal{Z}_N^{\beta, \underline{\mathbf{t}}} \geq e^{-\frac{3}{2}\beta n(\underline{\mathbf{t}})} \left\{ 2\rho(\Delta(\underline{\mathbf{t}})) + u^{\frac{3}{2}n(\underline{\mathbf{t}})} \sum_{k=2}^{n(\underline{\mathbf{t}})} \left(\frac{2}{u} \right)^k \sum_{\{\mathcal{C}_1, \dots, \mathcal{C}_k\} \in \pi(\underline{\mathbf{t}})} \prod_{i=1}^k f(\mathcal{C}_i) \right\}. \quad (3.16)$$

Implementation of the same technique as in the previous case yields the lower bound for the critical curve,

$$\mu < \frac{3}{2}\beta + \ln 2 + \frac{1}{2} \ln \left(1 - e^{-2\beta}\right) + \ln \left(1 + e^{-2\beta}\right) \quad \text{and} \quad \beta < \frac{\ln 2}{2}. \quad (3.17)$$

PROOF OF THEOREM 1. The proof of Theorem 1 follows immediately from Proposition 3 and the lower bound given by the Griffiths inequality. \square

Remark 2. *The lower bound given by (3.17) does not improve the lower bound given by the first Griffiths inequality.*

3.4 Proof of Theorem 2

For each $N \in \mathbb{N}$, we define the following sets in \mathbb{R}_+^2

$$\Gamma_N = \{(\beta, \mu) \in \mathbb{R}_+^2 : \mathbf{K}^N \text{ is a trace class in } \ell_{T-C}^2\}, \quad (3.18)$$

$$\Gamma^- = \bigcap_{N \in \mathbb{N}} \Gamma_N \quad \text{and} \quad \Gamma^+ = \bigcup_{N \in \mathbb{N}} \Gamma_N. \quad (3.19)$$

Note that $\Gamma^- \subset \Gamma_N \subset \Gamma^+$ for all $N \geq 1$, $\Gamma_N \uparrow \Gamma^+$, and that the finite-volume Gibbs measure $\mathbb{P}_N^{\beta, \mu}$ on the set Γ_N exists. We also define the N -strip functions ρ_N associated with the set Γ_N , for each $N \geq 1$, as follows

$$\rho_N(\beta) = \inf\{\mu \in \mathbb{R}_+^2 : (\beta, \mu) \in \Gamma_N\} \quad \text{for} \quad \beta \geq 0. \quad (3.20)$$

By the properties of the trace class operators (see [22] for an overview), we deduce that $\{\rho_N\}$ is a monotone decreasing sequence of measurable functions such that $f_1(\beta) \leq \rho_N(\beta)$, for all $\beta > 0$, where f_1 is defined in (2.28). Thus, we prove the existence of the pointwise limit

$$\rho_{T-C}(\beta) := \lim_{N \rightarrow \infty} \rho_N(\beta) \quad \text{for} \quad \beta \geq 0. \quad (3.21)$$

Another important fact is that the graph of the function ρ_{T-C} provides an upper bound for the critical curve. This property is a consequence of the spectral properties of the operator \mathbf{K} introduced in (2.13)(see [15] for the details and [22] for an overview) and the definition of the function ρ_{T-C} . Consequently, implementation the results of [15] and the condition of sub-criticality (2.21), we can prove the following proposition.

Proposition 4. *There exist $N_0 \in \mathbb{N}$ such that for $N > N_0$, the following property of functions f_N is fulfilled:*

1. *If $0 < \beta < \beta_2^*$, then*

$$f_N(\beta) \leq \psi(\beta), \quad (3.22)$$

where β_2^ is a positive solution of (2.27) and ψ is defined in (2.23).*

2. *If $\beta_2^* \leq \beta < \infty$, then*

$$f_N(\beta) \leq \frac{3}{2}\beta + 2 \ln 2. \quad (3.23)$$

Combining (3.21), (3.22), (3.23), and letting $N \rightarrow \infty$, we obtain the desired upper bound for the limit function f_{T-C}

$$\begin{cases} f_{T-C}(\beta) \leq \psi(\beta) & \text{if } 0 < \beta < \beta_2^* \\ f_{T-C}(\beta) \leq \frac{3}{2}\beta + 2 \ln 2 & \text{if } \beta_2^* \leq \beta < \infty. \end{cases} \quad (3.24)$$

Since that the graph of the function f_{T-C} lies above of the critical curve, the right-hand side of (3.24) provides an upper bound for the critical curve.

PROOF OF THEOREM 2. The upper bound for the critical curve γ_{cr} is a consequence of inequality (3.24). The lower bound is a consequence of Theorem 1. This concludes the proof of the theorem. \square

4 Discussion

In this article we present a step towards improving the subcriticality domain for an Ising model coupled to two-dimensional CDT introduced in [15]. In doing so we employ FK representation of the Ising model on causal triangulations and combinatorial approximation. In addition, we make a first step towards determining the critical curve of the model. Numerical evidence shows that the model has phase transition (see [3], [4], [8]), but this fact has never been proved explicitly. In this article we present mathematical evidence of existence of the critical curve by studying the infinite-volume free energy, and computing a region where the critical curve for the annealed model is possibly located (see Figure 2). The discussion in Section 2.2 along with Corollary 1 suggest that free energy can be expressed as $\ln \Lambda(\mu - \phi(\beta))$, where the function ϕ satisfies $\lim_{\beta \rightarrow 0^+} \phi(\beta) = \ln 2$. This result leads to a

plausible conjecture that the boundary of the subcritical domain coincides with the locus of points (β, μ) where $\mu = \phi(\beta) + \ln 2$, however in order to obtain more precise conclusions a further investigation is required.

In addition, Theorem 1 and Theorem 2 show that with respect to the weak limit Gibbs measure $\mathbb{P}^{\beta, \mu}$, the average diameter of any fixed height n is bounded in the region $\mu > f_2(\beta)$ (by a constant independent of n) with probability 1, which is essentially a one dimensional random geometry. Thus, we can expect that this subcritical behaviour can be extended to the region $\mu > \gamma_{cr}(\beta)$, and that any causal triangulation on the critical curve has Hausdorff dimension $d_H = 2$, $\mathbb{P}^{\beta, \mu_{cr}}$ -a.s. Finally, if the annealed model has this property, then the region where it presents a phase transition is possibly located on the critical curve. This direction also requires further research.

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